Non polynomial conservation law densities generated by the symmetry operators in some hydrodynamical models

Maxim V. Pavlov, Ziemowit Popowicz
University of Wrocław, Institute of Theoretical Physics
pl.M.Borna 9 50-205 Wrocław Poland
e-mail: ziemek@ift.uni.wroc.pl

February 8, 2008

Abstract

New extra series of conserved densities for the polytropic gas model and nonlinear elasticity equation are obtained without any references to the recursion operator or to the Lax operator formalism. Our method based on the utilization of the symmetry operators and allows us to obtain the densities of arbitrary homogeneity dimensions. The non polynomial densities with logarithmic behavior are presented as an example. The special attention is paid for the singular case ($\gamma = 1$) for which we found new non homogeneous solutions expressed in terms of the elementary functions.

Introduction

The conservation laws are the most important object in the classical mechanics as well as in the field theory. There are many different methods of the constructions of these laws. The most popular, especially used in the soliton theory and in the hydrodynamics, utilize the so called recursion operator or Lax operator formalism [1]. On the other hand, it appeared that for the Nonlinear Schroedinger equation it is possible to find one series of conserved Hamiltonians using recursion operator only. However for the shallow water equation, which is the dispersionless limit of the Nonlinear Schroedinger equation, there is additional series of conserved densities which is impossible to obtain by recursion operator (see for example [2]).

In this paper we would like to show that it is possible to construct new extra series of conserved densities for the polytropic gas model and nonlinear elasticity equation [3, 4] avoiding using recursion operator or Lax formalism. More precisely we generate many nonequivalent Hamiltonians of the given dimensions, using symmetry operator. Our Hamiltonians are non polynomial expressions which contains logarithmic functions. Independently we consider the singular case, for the polytropic gas system for which ($\gamma = 1$) . For this system we constructed new series of non homogeneous solutions expressed in terms of the elementary functions.

The paper is organized as follows. In the first section we describe the basic properties of the polytropic gas system which we use in the next sections. In the second section we describe our symmetry approach where we utilize the shift, scaling and projective operators in order to generate the conserved densities. In the third section we present explicitly new series of the conserved densities for different, physically interesting, models of the polytropic gas system with $\gamma = 2, 3, 4, 5, \frac{5}{3}, \frac{7}{5}, -1$. In the last section we adopt our formalism to the degenerated case ($\gamma = 1$) which is obtained from the contraction of the previous case. We show that in this case the conserved densities are connected with the Bessel equation.

1 The hydrodynamical systems.

The theory of the hydrodynamical type systems of the nonlinear equations [5]

$$u^i_t = \sum_{i=1}^{N} \upsilon^i_j(\mathbf{u}) u^j_x \quad i, j = 1, 2, ..., N \tag{1}$$

where $\mathbf{u} = (u_1, u_2, ...u_N)$ and υ^i_j are some functions , integrable by the generalized hodograph method [6] is closely related to the over-determined systems of first order partial differential linear equations. The conservation laws are such that

$$\frac{\partial h}{\partial t} = \frac{\partial g}{\partial x} \tag{2}$$

where h is density and g is flux. Then densities satisfy

$$\partial_k(\partial_i h) = \Gamma^i_{ik}(\partial_i h) + \Gamma^k_{ik}(\partial_k h). \quad i \neq k, \tag{3}$$

where

$$\Gamma^i_{ik} \equiv \frac{\partial_k \mu^i}{\mu^k - \mu^i} \quad i \neq k \quad \partial_k \equiv \partial/\partial r^k \tag{4}$$

and $r^k(\mathbf{u})$ are the so called Riemann invariants in which the hydrodynamic type system (1) is rewritten in the diagonal form

$$r^i_t = \mu^i(\mathbf{r}) r^i_x, \tag{5}$$

and no summation on the repeated indices. Thus, (3) is a linear systems of *first* order partial differential equations with variable coefficients. The general solution of such system is determined up to N arbitrary functions of a single variable.

There are many particular cases of the system (3) for which a general solution is expressed in explicit and in compact form ([7]). If we cannot to find a general solution of such system, then alternative way to solve the Cauchy or Goursat problems is to create the infinite number of particular solutions [8].

It appeared that the conserved densities can be used in the construction of particular solutions. Indeed. Let us consider the polytropic gas

$$\eta_t = \partial_x(u\eta), \quad u_t = \partial_x\Big[\frac{u^2}{2} + \frac{\eta^{\gamma-1}}{\gamma - 1}\Big], \tag{6}$$

and nonlinear elasticity equations

$$\eta_y = u_x, \quad u_y = \partial_x\Big[\frac{\eta^{\gamma-2}}{\gamma-2}\Big], \tag{7}$$

which are the commuting flows to each other.

The solution of the first system obtained by hodograph method is

$$t = h - \eta\frac{\partial h}{\partial \eta}, \quad x = -\frac{\partial h}{\partial \eta} - \frac{u}{\eta}\frac{\partial h}{\partial u} \tag{8}$$

while for the second is

$$x = \frac{\partial h}{\partial \eta}, \quad y = \frac{\partial h}{\partial u} \tag{9}$$

where h is some solution of the Tricomi like equation

$$h_{uu} = \eta^{3-\gamma}h_{\eta\eta}. \tag{10}$$

The previous equation is nothing but the equation on the conservation law densities for both systems. Formulas (8) and (9) realize the general or particular solution for both systems if h is the general or particular solution of the equation (10) respectively.

In [3] and [4] two infinite serieses of *quasi linear* conservation laws were constructed for the polytropic gas and for the nonlinear elasticity equations respectively. *Quasi linear* means, that these conservation law densities are polynomials with respect to u, η and η^γ where γ is an arbitrary polytropic constant.

However, as we see in the next section, these densities does not exhaust the set of all possible conserved densities.

2 Symmetry operator approach for $\gamma \neq 1$.

Let us first consider the more general form of the polytropic gas system

$$\eta_t = \partial_x(u\eta), \qquad u_t = \partial_x[\frac{u^2}{2} + \eta f''(\eta) - f'(\eta)], \tag{11}$$

$$\eta_y = u_x, \qquad u_y = \partial_x f''(\eta) \tag{12}$$

where f is an arbitrary function. Our generalized system constitute Hamiltonian equations with the following local structure

$$\eta_t = \partial_x\frac{\delta H}{\delta u}, \quad u_t = \partial_x\frac{\delta H}{\delta \eta}, \quad \eta_y = \partial_x\frac{\delta \tilde{H}}{\delta u}, \quad u_y = \partial_x\frac{\delta \tilde{H}}{\delta \eta}, \tag{13}$$

where

$$H = \int[\frac{1}{2}u^2\eta + \eta f'(\eta) - 2f(\eta)]dx,$$

$$\tilde{H} = \int[\frac{u^2}{2} + f'(\eta)]dx. \tag{14}$$

In order to obtain the conserved densities we try to eliminate the differentials dq and dp from the corresponding conservation laws

$$\partial_t h = \partial_x p, \qquad \partial_y h = \partial_x q \tag{15}$$

By direct calculation we have

$$dq = h_\eta du + f'''(\eta) h_u d\eta, \qquad dp = [u h_u + \eta h_\eta] du + [\eta f'''(\eta) h_u + u h_\eta] d\eta \tag{16}$$

The compatibility conditions ($(p_u)_\eta = (p_\eta)_u, \quad (q_u)_\eta = (q_\eta)_u$) lead to the Tricomi - like equation

$$h_{\eta\eta} = f'''(\eta) h_{uu}, \tag{17}$$

Since, this equation is compatible with the shift symmetry operator (see [10])

$$\delta = \partial / \partial u, \tag{18}$$

one can search solutions in the form

$$h_u = \lambda h, \tag{19}$$

where λ is an arbitrary parameter. This is the eigenvalue problem for the shift symmetry operator. Then Tricomi-like equation is transformed to the linear ordinary differential equation

$$h_{\eta\eta} = \lambda^2 f'''(\eta) h. \tag{20}$$

However this equation cannot be solved explicitly for an arbitrary f.

We propose, from that reason, quite different approach. Let us observe that the shift symmetry operator transforms one solution of the Tricomi - like equation onto another one

$$\partial_u h_{n+1} = h_n. \tag{21}$$

It means that all conservation law densities h_k and corresponding fluxes p_k, q_k can be written down in the quadratures recursively

$$dh_{k+1} = h_k du + q_k d\eta, \quad dq_{k+1} = q_k du + f'''(\eta) h_k d\eta, \tag{22}$$

$$dp_{k+1} = [u h_k + \eta q_k] du + [\eta f'''(\eta) h_k + u q_k] d\eta. \tag{23}$$

Interestingly all known examples of the conserved densities could be obtained in this way.

However we demonstrate different possibilities which give us new serieses of the conserved densities. Our main idea is based on the following observation.

The Tricomi like equation (10) is compatible with three local symmetry operators:

1. Shift operator

$$\delta = \frac{\partial}{\partial u}, \tag{24}$$

2. Scaling operator

$$R = u \frac{\partial}{\partial u} + \frac{2}{\gamma - 1} \eta \frac{\partial}{\partial \eta}, \tag{25}$$

3. Projective operator

$$S=[\frac{\gamma-1}{4}u^2+(\gamma-1)^{-1}\eta^{\gamma-1}]\frac{\partial}{\partial u}+u\eta\frac{\partial}{\partial\eta}+\frac{\gamma-3}{4}u. \tag{26}$$

These operators, with the identity operator, constitute closed Lie Algebra, with the following commutation relation

$$[\delta,S]=\frac{2}{\gamma-1}R+\frac{\gamma-3}{4},\qquad [\delta,R]=\delta,\qquad [R,S]=S \tag{27}$$

They act on *homogeneous* conservation law densities as follows

$$\delta h_{k+1}=h_k\quad Rh_k=c_kh_k\quad Sh_k=h_{k+1}, \tag{28}$$

where c_k are degree of homogeneity. Thus, by combination of these symmetry operators one can describe all quasi linear conservation law densities.

Interestingly if we rewrite the Tricomi -like equation using the Riemann invariants we obtain the famous Euler - Darboux - Poisson equation

$$\frac{\partial h}{\partial r_1\partial r_2}=\frac{\varepsilon}{r_1-r_2}\Big[\frac{\partial h}{\partial r_1}-\frac{\partial h}{\partial r_2}\Big], \tag{29}$$

where

$$\varepsilon=\frac{3-\gamma}{2(1-\gamma)},\quad r_1=u+\frac{2}{(\gamma-1)}\eta^{\frac{\gamma-1}{2}},\quad r_2=u-\frac{2}{(\gamma-1)}\eta^{\frac{\gamma-1}{2}}. \tag{30}$$

Now the symmetry operators are

$$\delta = \frac{\partial}{\partial r^1}+\frac{\partial}{\partial r^2},\quad R=r^1\frac{\partial}{\partial r^1}+r^2\frac{\partial}{\partial r^2}, \tag{31}$$

$$S = (r^1)^2\frac{\partial}{\partial r^1}+(r^2)^2\frac{\partial}{\partial r^2}+\varepsilon[r^1+r^2]. \tag{32}$$

The "zero–solutions" of the shift operator δ

$$\delta h_0=0 \tag{33}$$

can be found immediately

$$h_0^{(1)}=1,\quad h_0^{(2)}=\eta, \tag{34}$$

while for the projective operator one can obtain

$$h_1^{(1)}=u,\quad h_1^{(2)}=u\eta, \tag{35}$$

etc.

The "zero–solutions" of projective operator are easy to obtain also from Riemann invariant $(r^1,\, r^2)$ form

$$h=(r^1r^2)^{-\varepsilon}. \tag{36}$$

If we shift the Riemann invariants, in this formula, on arbitrary parameter λ we obtain the generating function on the conservation law densities. When $\lambda\to\infty$ we can obtain known densities.

In order to obtain new densities we demonstrate quite different possibilities . Let us consider the following recursive chain

$$Rh_k^{(0)} = c_k h_k^{(0)}, \quad Rh_k^{(1)} = c_k[h_k^{(1)} + h_k^{(0)}], \quad Rh_k^{(2)} = c_k[h_k^{(2)} + h_k^{(1)}] \tag{37}$$

etc..., where $h_k^{(0)}$ are the quasi linear conserved densities.

Now $h_k^{(i)}, i = 1, 2, ...$ are new conserved densities which satisfy the Tricomi-like equation (10). Notice that the second formula in (28) is not in contradiction to previous chain because formula (28) is valid for homogeneous conservation laws only.

Now let us consider the shallow water equation, e.g. the polytropic gas system with $\gamma = 2$. Let us choose the first homogeneous solutions of (28)

$$h_1^{(0)} = u, \qquad h_2^{(0)} = \eta, \qquad h_3^{(0)} = u\eta \qquad h_4^{(0)} = u^2\eta + \eta^2 \tag{38}$$

etc. As the result we obtained that

$$\begin{aligned}
h_1^{(1)} &= 2\sqrt{u^2 - 4\eta}\Big(\ln\Big(\sqrt{u^2 - 4\eta} + u\Big) - \ln 2\Big) + \Big(u - \sqrt{u^2 - 4\eta}\Big)\ln\eta \\
h_2^{(1)} &= \frac{u^2}{2} + \eta\ln\eta \\
h_3^{(1)} &= \frac{1}{4}\Big(u^3 + 6u\eta\ln\eta\Big) \\
h_4^{(1)} &= \frac{1}{6}\Big(u^4 + 12u^2\eta\ln\eta + 12\eta^2\ln\eta - 18\eta^2\Big)
\end{aligned} \tag{39}$$

are conserved density also. In the next let us apply this procedure to the $h_2^{(1)}$ and $h_3^{(1)}$

$$\begin{aligned}
h_2^{(2)} &= \frac{1}{18}\Big(- 36\eta\ln^2(\sqrt{z} + u) + 18\ln(\sqrt{z} + u)(2\eta u\ln(4\eta) - 2\eta + u\sqrt{z}) \\
&\quad +9u\sqrt{z}(1 - \ln(4\eta)) - 36\eta\ln\eta\ln 2 + (9u^2 + 18\eta)\ln\eta - 18u^2 + 32\eta\Big), \\
h_3^{(2)} &= \frac{1}{12}\Big(- 36\eta u\ln^2(\sqrt{z} + u) + \ln(\sqrt{z} + u)(36\eta u\ln(4\eta) - 6\eta u + \\
&\quad 6\sqrt{z}(u^2 + 8\eta)) - 36\eta u\ln(\eta)\ln 2 + 3\ln\eta(u^3 + 6\eta u - \sqrt{z}(u^2 + 8\eta)) + \\
&\quad 3\sqrt{z}(1 - 2\ln 2)(u^2 + 8\eta) - 16\eta u - 8u^3\Big),
\end{aligned} \tag{40}$$

where $z = u^2 - 4\eta$.

We obtained rather complicated formulas which contain the logarithm in the second power. Now we can continue this procedure obtaining complicated formulas which contain the logarithm in higher powers. On the other side we can combine our method with this which generate quasi linear densities obtaining following diagram

$$
\begin{tikzcd}
h_k^0 \arrow[r, "R"] \arrow[d, "S"'] & h_k^1 \arrow[r, "R"] \arrow[d] & h_k^2 \arrow[r] \arrow[d] & \cdots \\
h_{k+1}^0 \arrow[r, "R"] \arrow[d] & h_{k+1}^1 \arrow[r, "R"] \arrow[d] & h_{k+1}^2 \arrow[r] \arrow[d] & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{tikzcd}
$$

This diagram obviously is closed. In order to see it let us start consideration from some $h_k^{(n)}$. Applying at first projective and at next step scaling transformations we obtain

$$Rh_{k+1}^{(n+1)} = c_{k+1}\Big(h_{k+1}^{(n+1)} + Sh_k^{(n)}\Big) \tag{41}$$

Finally applying these operators in the reverse order we obtain

$$SRh_k^{(n+1)} = c_k\Big(Sh_k^{(n+1)} + Sh_k^{(n)}\Big) \tag{42}$$

Using the commutation relation (27) we see that these formulas coincide because $c_{k+1} = c_k + 1$.

3 Non polynomial densities for an arbitrary $\gamma \neq 1$ in polytropic gas system

For an arbitrary γ our generating equations become

$$Rh_k^{(n)} = c_k\Big(h_k^{(n)} + h_k^{(n-1)}\Big). \tag{43}$$

Now we have to solve the Tricomi like equation on $h_k^{(n)}$ which is however rather hard to do. We restrict, from that reason, to two cases where $h_1^{(0)} = u$ and $h_2^{(0)} = \eta$.

For the first case we have $c_1 = 1$. By direct verification one can check that

$$h_1^{(1)} = uh(\tau) + u \ln u \tag{44}$$

where $\tau = \frac{u^2}{\eta^{\gamma-1}}$ satisfy (43). The Tricomi like equation reduces in this case to

$$\tau^2\Big(4 - (\gamma-1)^2\Big)\frac{\partial^2 h}{\partial^2 \tau} + \Big(6 - \gamma(\gamma-1)\tau\Big)\frac{\partial h}{\partial \tau} + 1 = 0. \tag{45}$$

For the second case we have $c_2 = \frac{2}{\gamma-1}$. Similarly to the first case, by direct verification, one can check that

$$h_2^{(1)} \quad = \quad \eta h(\tau) + \eta \ln \eta \tag{46}$$

satisfy (43). Substituting $h_2^{(1)}$ to the Tricomi like equation (10) we obtain the following equation on the function $h(\tau)$

$$\tau\Big(4-\tau(\gamma-1)\Big)\frac{\partial^2 h}{\partial^2\tau}-\Big(2-(\gamma-1)(\gamma-2)\tau\Big)\frac{\partial h}{\partial\tau}+1=0 \tag{47}$$

In both cases it is possible to obtain closed formulas for particular values of γ parameter. However we can simplify our consideration using different coordinates. For example choosing the coordinates r and p as [3]

$$\eta=rp,\qquad u=\frac{1}{\gamma-1}\Big(r^{\gamma-1}+p^{\gamma-1}\Big), \tag{48}$$

we can rewrite the scaling symmetry operator and Tricomi like equation as

$$\begin{aligned} R &= \frac{1}{\gamma-1}\Big(r\frac{\partial}{\partial r}+p\frac{\partial}{\partial p}\Big)\\ p^{3-\gamma}\frac{\partial^2 h}{\partial^2 p} &= r^{3-\gamma}\frac{\partial^2 h}{\partial^2 r} \end{aligned} \tag{49}$$

Taking into an account $h_1^{(0)}=r$ we obtained the following solutions on the $h_1^{(1)}$ densities

$$\begin{aligned} \gamma=2,\qquad h_1^{(1)} &= -(r-p)\ln(r-p)+p\ln(p)\\ \gamma=3,\qquad h_1^{(1)} &= (r+p)\ln(r+p)-(r-p)\ln(r-p)\\ \gamma=4,\qquad h_1^{(1)} &= -(r-p)\ln(r-p)+(r+2p)\ln(p^2+pr+r^2)\\ &\quad+\sqrt{3}p\arctan(\frac{p+2r}{\sqrt{3}p})\\ \gamma=5,\qquad h_1^{(1)} &= (p+r)\ln(p+r)+(r-p)\ln(r-p)+r\ln(p^2+r^2)\\ &\quad+2p\arctan(\frac{r}{p}) \end{aligned} \tag{50}$$

Finally let us present the solution for $\gamma=\frac{5}{3},\frac{7}{5}$ and $\gamma=-1$ which are interesting from the physical point because these describe dynamics of one-atomic gas and two-atomic gas [11] and two dimensional nonlinear Born-Infeld electrodynamics [12] respectively.

$$\begin{aligned} \gamma=\frac{5}{3},\qquad & h_1^{(1)}=\frac{3}{4}p\ \mathrm{arctanh}\tau^{\frac{1}{3}}+\frac{r}{2}\ln\big(3\tau^{\frac{2}{3}}-3\tau^{\frac{4}{3}}+\tau^2-1\big)\\ & \quad -r\ln\tau-3r\tau^{-\frac{2}{3}}\\ \gamma=\frac{7}{5},\qquad & h_1^{(1)}=\frac{5}{4}p\ \mathrm{arctanh}\tau^{\frac{1}{5}}+\frac{r}{2}\ln\big(-10\tau^{\frac{4}{5}}-5\tau^{\frac{8}{5}}+5\tau^{\frac{2}{5}}+\tau^{\frac{6}{5}}+\tau^2-1\big)\\ & \quad -r\ln\tau-\frac{5}{3}r\tau^{-\frac{2}{5}}-5r\tau^{-\frac{4}{5}}+r\ln r\\ \gamma=-1,\qquad & h_1^{(1)}=\frac{1}{2}\Big(r\ln\big(\frac{p^2r^2}{r^2-p^2}\big)+p\ln\big(\frac{r-p}{r+p}\big)\Big) \end{aligned} \tag{51}$$

4 Symmetry operator approach for $\gamma = 1$ and corresponding non polynomial densities.

For this case the polytropic gas and the nonlinear elasticity systems are

$$\eta_t = \partial_x(u\eta), \quad u_t = \partial_x\Big[\frac{u^2}{2} + \ln\eta\Big]. \tag{52}$$

$$\eta_y = u_x, \quad u_y = \partial_x(-\frac{1}{\eta}), \tag{53}$$

respectively. For this case the Euler - Darboux - Poisson equation (29) degenerates because $\varepsilon \to \infty$. Similarly the scaling operator R (see (25)) also becomes degenerated.

In order to solve this problem we notice that the analogue of the symmetry algebra (26) can be obtained considering the contraction with respect to $\gamma = 1$. If we rescale $R \to \frac{\gamma-1}{2}R$ and compute the limit when $\gamma \to 1$ we obtain

$$\delta = \partial_u, \quad R = \eta\partial_\eta, \quad S = \ln\eta\partial_u + u\eta\partial_\eta - u/2, \tag{54}$$

Interestingly now, in the contraction limit, the homogeneity properties (28) does not hold. It means that the Tricomi-like equation

$$h_{uu} = \eta^2 h_{\eta\eta} \tag{55}$$

has no any homogeneous solutions.

However the quasi-homogeneous solutions could be obtained ([9]). By quasi-homogeneous we understand the homogeneous with respect to the one variable u or $\ln\eta$.

For our further applications we describe another possibility. We constructed the generating function for the conserved densities. To end this we consider the eigenvalue problem for each symmetry operator.

1. For the scaling operator R (see (54)) the eigenvalue problem

$$h_\xi = \lambda h, \tag{56}$$

where $\partial_\xi \equiv \eta\partial_\eta$, reduces the Tricomi-like equation to the linear ordinary differential equation of the second order

$$h_{uu} = \lambda(\lambda - 1)h. \tag{57}$$

These equations can be easily integrated and we obtained

$$h = \exp[\sqrt{\lambda(\lambda-1)}u + \lambda\xi]. \tag{58}$$

2. For the shift operator δ (see (54)) the eigenvalue problem

$$h_u = \tilde{\lambda}h \tag{59}$$

reduces the Tricomi-like equation to the linear ordinary differential equation of the second order

$$h_{\xi\xi} - h_\xi = \tilde{\lambda}^2 h, \tag{60}$$

These equation are easily also to integrate and the solutions have the same form as (58) in which we should replace $\lambda^2 \to \tilde{\lambda}(\tilde{\lambda}-1)$.

If one expands function h near $\lambda \to 0$ or $\lambda \to 1$, then one can obtain two well-known infinite serieses of conservation law densities ([9]).

3. Projective operator S (see (54)). If one substitutes

$$h = q\exp[\xi/2], \quad u = 2s\cosh\theta, \quad \xi = 2s\sinh\theta, \tag{61}$$

where s and θ are new functions, then the eigenvalue problem looks like

$$q_\theta = \lambda q \tag{62}$$

Now the generating function is

$$q = \psi(s)\exp[\lambda\theta], \tag{63}$$

where the function ψ satisfies the Bessel equation

$$\psi'' + \frac{1}{s}\psi' + [1 - \frac{\lambda^2}{s^2}]\psi = 0, \tag{64}$$

Let us present the most simplest example of the conserved densities obtained in this way, where it is possible to describe the Bessel function by the elementary functions for which $\lambda = \frac{1}{2}$

$$h = \sqrt{\frac{\eta}{u - \ln\eta}}\cos\Big(\frac{\sqrt{u^2 - \ln^2\eta}}{2}\Big). \tag{65}$$

Using the recursion relations, known in the theory of Bessel functions, it is possible to generate infinite sets of conserved densities written in terms of elementary functions without any references to the recursion operator, which appear in the Hamiltonian approach to the polytropic gas systems. Finally we can use the shift or scaling symmetry operators and generate some conserved densities as we did in the previous sections.

5 Conclusion

In this paper we constructed new extra series of conserved densities for the polytropic gas model and nonlinear elasticity equation avoiding using recursion operator or Lax formalism. Our Hamiltonians appeared as the non polynomial expressions which contain logarithmic functions. If we continue our procedure to these logarithmic densities in the next step we obtain expressions with the logarithmic function in an arbitrary power. We considered the singular case of the polytropic gas system also, for which we found the non homogeneous solutions expressed in terms of Bessel functions. If we apply the point transformation to the symmetry operators then they can change the role. It means that the scaling and shift operators are equivalent to each other, what it easy to see in the Riemann invariants (31) or (59,56). We presented this method for two hydrodynamical systems only, but this method can be adopted to more complicated equations as well.

Acknowledgments

One of the author (M.P.) thanks the Institute of Theoretical Physics University of Wrocław for their hospitality.

References

[1] M.Błaszak ,,*Multi-Hamiltonian Theory of Dynamical Systems*" Springer -Verlag 1998, P.Olver ,, *Applications of Lie Group to Differential Equations*" , Springer - Verlag 1989.

[2] M. V. Pavlov, S. P. Tsarev, Uspekhi Mat. Nauk, 46, No. 4, 169–170 (1991).

[3] Y.Nutku, M.Pavlov J.Math.Phys **43** (2002) 1441.

[4] J.C.Brunelli, A.Das Phys.Lett **B** 426 (1998) 57.

[5] B. A. Dubrovin, S. P. Novikov, Dokl. Akad. Nauk SSSR, 270, No. 4, (1993) 781.

[6] S. P.Tsarev, Dokl. Akad. Nauk SSSR, 282, No. 3, 534–537 (1985).

[7] M.Pavlov ,, *Integrable hydrodynamic chains*" nlin.SI/00301010.

[8] S.P.Tsarev Izv. Akad. Nauk SSSR, ser. Matem., 54, No. 5, 1048–1068 (1990).

[9] Y.Nutku J.Math.Phys **28** (1987) 2579.

[10] M. V. Pavlov, S.P. Tsarev Funct. Anal.Appl **37** No.1 (2003) 38.

[11] B.L.Rozhdestvenski, N.N. Yanenko ,, *Systems of quasilinear equations and their applications to gas dynamics*" Mathematical Monographs, **55** AMS, Providence, RI (1983).

[12] M.Arik, F.Neyzi, Y.Nutku, P.J.Olver, J.Verosky J.Math.Phys **30** (1988) 1338.